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Superposition operators between Q_p spaces and Hardy spaces ☆

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ABSTRACT

For any pair of numbers (s, p) with $0 \leqslant s < \infty$ and $0 < p \leqslant \infty$ we characterize the superposition operators which apply the conformally invariant Q_s space into the Hardy space H^p and, also, those which apply H^p into Q_s.

1. Introduction and results

We denote by $\mathbb{D}$ the unit disc $\{z \in \mathbb{C}\colon |z| < 1\}$ and by $\mathbb{T}$ the unit circle $\{\xi \in \mathbb{C}\colon |\xi| = 1\}$. The space of all analytic functions on $\mathbb{D}$ will be denoted by $\mathcal{H}ol(\mathbb{D})$.

Given an entire function φ, the superposition operator $S_\varphi : \mathcal{H}ol(\mathbb{D}) \to \mathcal{H}ol(\mathbb{D})$ is defined by $S_\varphi(f) = \varphi \circ f$.

If X and Y are two subspaces of $\mathcal{H}ol(\mathbb{D})$, the question is: For which entire functions φ does the operator S_φ map X into Y?

This question has been studied for distinct pairs (X, Y) of spaces of analytic functions, see, e.g., [1,6–10,20,21]. Let us just mention that Buckley and Vukotić proved the following result (see Theorem 1 in the case $p = 2$ and Remark 8 of [8]):

Theorem A. *Let $0 < p < \infty$. If φ is an entire function, then S_φ maps the Dirichlet space $\mathcal{D}$ into the Hardy space H^p if and only if φ is of order less than 2, or of order 2 and finite type.*

We refer to [15, Chapter 14] for all the facts regarding the order and type of an entire function needed in the paper.

Our main purpose in this paper is to characterize the superposition operators which apply Q_s spaces $(0 \leqslant s < \infty)$ into H^p spaces $(0 < p \leqslant \infty)$.

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If $0 \leqslant s < \infty$, we say that $f \in Q_s$ if f is analytic in $\mathbb{D}$ and

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^2 g(z,a)^s \, dx\, dy < \infty,$$

where $g(z,a)$ is the Green's function in $\mathbb{D}$, given by $g(z,a) = \log\left|\frac{1-\bar{a}z}{z-a}\right|$.

These spaces were introduced by Aulaskari and Lappan in [2] while looking for new characterizations of Bloch functions. They proved that for $s > 1$, Q_s is the Bloch space $\mathcal{B}$ which consists of those functions f analytic in $\mathbb{D}$ for which

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)\left|f'(z)\right| < \infty.$$

Using one of the many characterizations of the space *BMOA* (see, e.g., Theorem 5 of [5]) we see that $Q_1 = BMOA$. If $s = 0$, Q_s is the classical Dirichlet space $\mathcal{D}$ of those analytic functions f in $\mathbb{D}$ satisfying $\int_{\mathbb{D}} |f'(z)|^2 \, dx\, dy < \infty$.

It is well known that $\mathcal{D} \subset BMOA$ and $BMOA \subset \mathcal{B}$. Aulaskari, Xiao and Zhao proved in [4] that

$$\mathcal{D} \subset Q_{s_1} \subset Q_{s_2} \subset BMOA, \quad 0 < s_1 < s_2 < 1.$$

We mention the books [18,19] as general references for the theory of Q_s-spaces.

Our main result is the following.

Theorem 1. *Let φ be an entire function. Then*

(a) *For $0 < s \leqslant 1$ and $0 < p < \infty$, $S_\varphi(Q_s) \subset H^p$ if and only if φ is of order less than one, or of order one and type zero.*
(b) *For $0 < p < \infty$, $S_\varphi(\mathcal{B}) \subset H^p$ if and only if φ is constant.*
(c) *For $0 \leqslant s < \infty$, $S_\varphi(Q_s) \subset H^\infty$ if and only if φ is constant.*

Let us remark that putting together Theorem A and Theorem 1 we have a complete characterization of the superposition operators mapping Q_s-spaces into H^p-spaces for all the cases ($0 \leqslant s < \infty$, $0 < p \leqslant \infty$).

We shall also characterize the superposition operators which apply Hardy spaces into Q_s-spaces, as follows.

Theorem 2. *Let φ be an entire function. Then*

(a) *For $0 < p < \infty$ and $0 \leqslant s < \infty$, $S_\varphi(H^p) \subset Q_s$ if and only if φ is constant.*
(b) *For $0 \leqslant s < 1$, $S_\varphi(H^\infty) \subset Q_s$ if and only if φ is constant.*
(c) *For $1 \leqslant s < \infty$, $S_\varphi(H^\infty) \subset Q_s$.*

2. Background

In this section we define all function spaces and classes which will be considered later, as well as certain concepts, and fix the notation.

2.1. Hardy spaces and the Nevanlinna class

If $0 \leqslant r < 1$ and f is a function which is analytic in $\mathbb{D}$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p}, \quad \text{if } 0 < p < \infty,$$

$$M_\infty(r,f) = \max_{|z|=r} \left|f(z)\right|.$$

For $0 < p \leqslant \infty$, the Hardy space H^p consists of those functions f, analytic in $\mathbb{D}$, for which

$$\|f\|_{H^p} = \sup_{0\leqslant r<1} M_p(r,f) < \infty.$$

Then we see that H^∞ is the class of bounded analytic functions in the disc. It is clear that $H^p \supset H^q$, whenever $0 < p < q \leqslant \infty$.

The Nevanlinna class $\mathcal{N}$ is defined as the set of all analytic functions f in $\mathbb{D}$ such that the integrals $\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| \, d\theta$ are bounded for $0 \leqslant r < 1$. It is easy to check that $\mathcal{N} \supset H^p$ for every $p > 0$. We refer to [11] for the theory of H^p spaces.

For each function $f \in \mathcal{N}$, the radial limit $\lim_{r\to 1} f(re^{i\theta})$ exists almost everywhere. If in addition $f \in H^p$ for some $p > 0$, then the function $e^{i\theta} \mapsto f(e^{i\theta})$ of the boundary values of f belongs to $L^p(\mathbb{T})$.

We will use also the following well-known property: If $f \in \mathcal{N}$, and f does not vanish identically, then the sequence $\{z_n\}$ of its zeros, repeated according to multiplicity, must satisfy the Blaschke condition, that is, $\sum(1-|z_n|) < \infty$.

2.2. The space BMOA

If E is a (Lebesgue) measurable subset of $\mathbb{T}$, the (one-dimensional) Lebesgue measure of E will be denoted by $|E|$, hence $|\mathbb{T}| = 2\pi$.

If $f \in L^1(\mathbb{T})$ and I is an interval of $\mathbb{T}$ with $|I| > 0$, let $m_I(f)$ denote the mean of f over the interval I, that is, $m_I(f) = \frac{1}{|I|}\int_I f$.

The mean oscillation of f over I is

$$m_I\big(|f - m_I(f)|\big) = \frac{1}{|I|}\int_I |f - m_I(f)|.$$

Taking the supremum of these quantities as I ranges over all intervals I of $\mathbb{T}$ (with $|I| > 0$), we define

$$\|f\|_* = \sup_I\left(\frac{1}{|I|}\int_I |f - m_I(f)|\right).$$

We say that f has bounded mean oscillation or that $f \in BMO$ if these supremum is finite, that is:

$$BMO = \big\{f \in L^1(\mathbb{T}):\ \|f\|_* < \infty\big\}.$$

We refer to [5,13] as general sources for the space BMO. The inclusion $L^\infty(\mathbb{T}) \subset BMO$ is obvious. This inclusion is strict. For example, the unbounded function $f(e^{i\theta}) = \log|\theta|$, $-\pi \leqslant \theta \leqslant \pi$, belongs to BMO. John and Nirenberg proved in [16] the following theorem, which shows that, even though there exist unbounded functions in BMO, they cannot grow too fast.

Theorem B. *There exist two positive constants K and β such that, whenever $f \in BMO$ and I is an interval of $\mathbb{T}$ with $|I| > 0$, we have*

$$\frac{1}{|I|}\big|\big\{e^{i\theta} \in I:\ \big|f\big(e^{i\theta}\big) - m_I(f)\big| > \lambda\big\}\big| \leqslant Ke^{-\beta\frac{\lambda}{\|f\|_*}}, \quad \textit{for every } \lambda > 0.$$

The space $BMOA$ consists of those functions $f \in H^1$ such that the function $e^{i\theta} \mapsto f(e^{i\theta})$ of the boundary values of f belongs to BMO. If $f \in BMOA$, we denote $\|f\|_* = \|f(e^{i\theta})\|_*$. It is clear that $H^\infty \subset BMOA$.

2.3. Univalent functions and hyperbolic metric

A univalent function in $\mathbb{D}$ is an analytic function which is one-to-one in the disc. By the Riemann mapping theorem, for any given simply connected domain Ω (other than the plane itself) there is such a function f (called a Riemann map) that takes $\mathbb{D}$ onto Ω and the origin to a prescribed point. Denoting by $\operatorname{dist}(w, \partial\Omega)$ the Euclidean distance of the point w to the boundary of the domain Ω, the Riemann map f has the following property (see Corollary 1.4 of [17], for example):

$$\frac{1}{4}\big(1-|z|^2\big)\big|f'(z)\big| \leqslant \operatorname{dist}\big(f(z), \partial\Omega\big) \leqslant \big(1-|z|^2\big)\big|f'(z)\big|, \quad z \in \mathbb{D}. \tag{1}$$

We will need a few basic properties of the hyperbolic metric. Recall that the hyperbolic distance between two points z_1 and z_2 in the disc is defined as

$$\rho(z_1, z_2) = \inf_\gamma \int_\gamma \frac{|d\zeta|}{1-|\zeta|^2} = \frac{1}{2}\log\frac{1 + |\frac{z_1 - z_2}{1-\bar{z}_1 z_2}|}{1 - |\frac{z_1 - z_2}{1-\bar{z}_1 z_2}|},$$

where the infimum is taken over all rectifiable curves γ in $\mathbb{D}$ that join z_1 with z_2.

The hyperbolic metric ρ_Ω on an arbitrary simply connected domain Ω (not the entire plane) is defined via the corresponding pullback to the disc: if f is a Riemann map of $\mathbb{D}$ onto Ω, then

$$\rho_\Omega\big(f(z_1), f(z_2)\big) = \rho(z_1, z_2) = \inf_\Gamma \int_{f^{-1}(\Gamma)} \frac{|d\zeta|}{1-|\zeta|^2}, \quad z_1, z_2 \in \mathbb{D},$$

where the infimum is taken over all rectifiable curves Γ in Ω from $f(z_1)$ to $f(z_2)$. The metric ρ_Ω does not depend on the choice of the Riemann map f. For more details we refer the reader to Sections 1.2 and 4.6 of [17].

From the definition of the hyperbolic metric we deduce the following important feature of Riemann maps: if $f(0)=0$ then

$$\rho_\Omega\big(0, f(z)\big) = \rho(0,z) \geqslant \frac{1}{2}\log\frac{1}{1-|z|}, \quad z \in \mathbb{D}. \tag{2}$$

Another fundamental property, which is easily deduced from (1), is that

$$\rho_\Omega(w_1, w_2) \leqslant \inf_\Gamma \int_\Gamma \frac{|dw|}{\operatorname{dist}(w, \partial\Omega)}, \quad w_1, w_2 \in \Omega, \tag{3}$$

where the infimum is taken over all rectifiable curves Γ in Ω from w_1 to w_2.

3. Proof of Theorem 1

We shall use the following lemma in our proof of Theorem 1.

Lemma 3. *If $f \in BMOA$, for $0<r<1$ define $f_r(z)=f(rz)$, $z\in\mathbb{D}$. Then $\|f_r\|_* \leqslant 3\|f\|_*$.*

Proof. Notice that $f_r \in H^\infty$, and then $f_r \in BMOA$. For each $t\in\mathbb{R}$, we denote $T_t f(e^{i\theta}) = f(e^{i(\theta-t)})$, $\theta\in\mathbb{R}$. Observe that $T_t f$ belongs to $L^1(\mathbb{T})$ and $\|T_t f\|_* = \|f\|_*$. Since $f\in H^1$, for all $\theta\in\mathbb{R}$ we can write

$$f\big(re^{i\theta}\big) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t) f\big(e^{i(\theta-t)}\big)\,dt,$$

where $P_r(t) = \frac{1-r^2}{|1-re^{it}|^2}$ denotes the Poisson kernel (see Theorem 3.1 of [11]).

We have

$$\|f_r - f\|_* = \sup_I \left(\frac{1}{|I|}\int_I \big|f_r\big(e^{i\theta}\big) - f\big(e^{i\theta}\big) - m_I(f_r - f)\big|\,d\theta\right), \tag{4}$$

where the supremum is taken over all the intervals I of $\mathbb{T}$ with $|I|>0$. For each interval I, the mean of $f_r - f$ over I can be expressed in the form

$$\begin{aligned}
m_I(f_r - f) &= \frac{1}{|I|}\int_I \big(f\big(re^{is}\big) - f\big(e^{is}\big)\big)\,ds \\
&= \frac{1}{|I|}\int_I \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t) f\big(e^{i(s-t)}\big)\,dt - f\big(e^{is}\big)\right) ds \\
&= \frac{1}{|I|}\int_I \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\big[f\big(e^{i(s-t)}\big) - f\big(e^{is}\big)\big]\,dt\right) ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\frac{1}{|I|}\int_I \big[T_t f\big(e^{is}\big) - f\big(e^{is}\big)\big]\,ds\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t) m_I(T_t f - f)\,dt,
\end{aligned}$$

and representing the first term $f_r(e^{i\theta})$ in (4) also in terms of the Poisson kernel, we obtain

$$\begin{aligned}
\frac{1}{|I|}\int_I \big|f_r\big(e^{i\theta}\big) - f\big(e^{i\theta}\big) - m_I(f_r - f)\big|\,d\theta &= \frac{1}{|I|}\int_I \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\big[f\big(e^{i(\theta-t)}\big) - f\big(e^{i\theta}\big) - m_I(T_t f - f)\big]\,dt\right| d\theta \\
&= \frac{1}{|I|}\int_I \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\big[T_t f\big(e^{i\theta}\big) - f\big(e^{i\theta}\big) - m_I(T_t f - f)\big]\,dt\right| d\theta
\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{1}{|I|}\int_I \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\left|T_t f\left(e^{i\theta}\right) - f\left(e^{i\theta}\right) - m_I(T_t f - f)\right| dt\, d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\frac{1}{|I|}\int_I \left|T_t f\left(e^{i\theta}\right) - f\left(e^{i\theta}\right) - m_I(T_t f - f)\right| d\theta\, dt \\
&\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\|T_t f - f\|_*\, dt \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\left(\|T_t f\|_* + \|f\|_*\right) dt \\
&= 2\|f\|_* \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\, dt = 2\|f\|_*.
\end{aligned}$$

Then, $\|f_r - f\|_* \leqslant 2\|f\|_*$, and, consequently,

$$\|f_r\|_* \leqslant \|f_r - f\|_* + \|f\|_* \leqslant 2\|f\|_* + \|f\|_* = 3\|f\|_*. \qquad \square$$

Proof of Theorem 1(a). The proof of the first implication in this case can be obtained adapting the argument in the proof of (a) $\Rightarrow$ (c) in [1, Theorem 3] because the Bloch function used there is univalent, and then it also belongs to the space Q_s. The difference is that one has to use the corresponding standard estimate for the Hardy spaces: if $g \in H^p$ $(0 < p < \infty)$, then

$$\left|g(z)\right| \leqslant \frac{\|g\|_{H^p}}{(1 - |z|^2)^{1/p}}, \quad z \in \mathbb{D}. \tag{5}$$

Thus, suppose that $0 < s \leqslant 1$, $0 < p < \infty$ and that S_φ maps Q_s into H^p. Suppose also that φ has order greater than one, or that it has order one and type different from zero, to get a contradiction. Then there exist $\varepsilon > 0$ and a sequence of complex numbers $\{w_n\}_{n=1}^{\infty}$ such that $|w_n| \to \infty$, as $n \to \infty$, and

$$\left|\varphi(w_n)\right| \geqslant e^{\varepsilon |w_n|}, \quad \text{for all } n. \tag{6}$$

Let us fix a positive constant δ such that $\delta > 6/\varepsilon p$. We can now choose a subsequence, denoted again $\{w_n\}_{n=1}^{\infty}$, so that $w_n \neq 0$ for all n, the sequence $\{\arg w_n\}_{n=1}^{\infty}$ in $[0, 2\pi]$ is convergent, and all points w_n lie in an angular sector of opening $\frac{\pi}{2}$. We may further assume with no loss of generality that they are all located in the first quadrant and the arguments $\arg w_n$ decrease to zero, by applying symmetries or rotations if necessary. Observe that the entire functions ψ and φ_t, defined by $\psi(z) = \overline{\varphi(\bar{z})}$ and $\varphi_t(z) = \varphi(e^{it}z)$, respectively, have the same order and type as φ.

Now, let $w_0 = 0$, and select inductively a further subsequence, labeled again $\{w_n\}_{n=1}^{\infty}$, in such a way that $|w_1| \geqslant 5\delta$, and

$$|w_n| \geqslant \max\left\{3|w_{n-1}|, \sum_{k=1}^{n-1}|w_k - w_{k-1}|\right\}, \quad \text{for all } n \geqslant 2. \tag{7}$$

Next, use Lemma 2 of [1] to construct a domain Ω in $\mathbb{C}$ with the following properties:

(i) Ω is simply connected,
(ii) Ω contains the infinite polygonal line $L = \bigcup_{n=1}^{\infty}[w_{n-1}, w_n]$, where $[w_{n-1}, w_n]$ denotes the line segment from w_{n-1} to w_n,
(iii) any Riemann map f of $\mathbb{D}$ onto Ω belongs to $\mathcal{B}$,
(iv) $\operatorname{dist}(w, \partial\Omega) = \delta$, for each point w on L.

Let f be a Riemann map of $\mathbb{D}$ onto Ω such that $f(0) = 0$. Now, f is a univalent function in $\mathcal{B}$, so we have that $f \in Q_s$ (see Theorem 6.1 and Remark 9 of [3]). Then, by assumption, we know that $S_\varphi f = \varphi \circ f \in H^p$. Now let z_n be the points in $\mathbb{D}$ for which $w_n = f(z_n)$. Since $|w_n| \to \infty$, as $n \to \infty$, it follows that $|z_n| \to 1$. By applying estimate (2) for the hyperbolic metric, the triangle inequality, inequality (3), property (iv) from Lemma 2 used above, and the properties (7) of the points w_n, respectively, we obtain the following chain of inequalities:

$$\begin{aligned}
\frac{1}{2}\log\frac{1}{1 - |z_n|} &\leqslant \rho_\Omega(0, w_n) \leqslant \sum_{k=1}^{n}\rho_\Omega(w_{k-1}, w_k) \leqslant \sum_{k=1}^{n}\int_{[w_{k-1}, w_k]}\frac{|dw|}{\operatorname{dist}(w, \partial\Omega)} \\
&= \sum_{k=1}^{n}\int_{[w_{k-1}, w_k]}\frac{|dw|}{\delta} = \frac{1}{\delta}\sum_{k=1}^{n}|w_k - w_{k-1}| \leqslant \frac{3}{\delta}|w_n|.
\end{aligned}$$

This shows that

$$|w_n| \geqslant \frac{\delta}{6}\log\frac{1}{1-|z_n|}. \tag{8}$$

It follows from (6) and (8) that

$$\left|\varphi(w_n)\right| \geqslant \exp\left(\frac{\varepsilon\delta}{6}\log\frac{1}{1-|z_n|}\right) = \frac{1}{(1-|z_n|)^{\varepsilon\delta/6}}, \quad \text{for all } n. \tag{9}$$

On the other hand, we know that $\varphi \circ f \in H^p$, then by (5), we have

$$\left|\varphi(w_n)\right| = \left|(\varphi\circ f)(z_n)\right| \leqslant \frac{\|\varphi\circ f\|_{H^p}}{(1-|z_n|^2)^{1/p}} \leqslant \frac{\|\varphi\circ f\|_{H^p}}{(1-|z_n|)^{1/p}}, \quad \text{for all } n. \tag{10}$$

However, (9) and (10) contradict each other since $\frac{\varepsilon\delta}{6} > \frac{1}{p}$ by our initial choice of δ. This proves that φ is of order less than one, or of order one and type zero.

Let us turn now to the other implication. Hence, suppose that φ is an entire function of order less than one, or of order one and type zero and let $0 < s \leqslant 1$ and $0 < p < \infty$. We are going to prove that $S_\varphi(Q_s) \subset H^p$. Take $f \in Q_s$. We may assume that f is non-constant (if f is constant then $S_\varphi f = \varphi\circ f$ is constant, so it is clear that $S_\varphi f \in H^p$). We may also assume that $f(0) = 0$. Otherwise we can take the function $g = f - f(0)$. Then $g \in Q_s$ and $g(0) = 0$. Define $\psi(z) = \varphi(z + f(0))$, $z \in \mathbb{C}$, entire function of the same order and type of φ. Then $S_\psi g$ belongs to H^p, and this function is

$$S_\psi g(z) = \psi\big(g(z)\big) = \varphi\big(g(z) + f(0)\big) = \varphi\big(f(z)\big), \quad z \in \mathbb{D}.$$

Then, let us suppose that f is a non-constant function in Q_s with $f(0) = 0$. Observe that $f \in Q_1 = BMOA$ and $0 < \|f\|_* < \infty$. Now take the absolute constants $K, \beta > 0$ given by the John–Nirenberg theorem, and fix ε such that

$$0 < \varepsilon < \frac{\beta}{3p\|f\|_*}. \tag{11}$$

Since φ is of order less than one, or of order one and type zero, there exists $R > 0$ such that

$$\left|\varphi(z)\right| \leqslant e^{\varepsilon|z|}, \quad \text{if } |z| \geqslant R.$$

Let $A = \max_{|z|\leqslant R}|\varphi(z)|$, and $B = \max(A, 1)$. Then,

$$\left|\varphi(z)\right| \leqslant Be^{\varepsilon|z|}, \quad \text{for all } z \in \mathbb{C}. \tag{12}$$

Given $0 < r < 1$, define $f_r(z) = f(rz)$, $z \in \mathbb{D}$. Then f_r is a non-constant function in H^∞, so $f_r \in BMOA$ and $0 < \|f_r\|_* < \infty$. Let us remark that the function $e^{i\theta} \mapsto f_r(e^{i\theta}) = f(re^{i\theta})$ of the boundary values of f_r belongs to $L^\infty(\mathbb{T})$ and hence to BMO.

We have

$$\begin{aligned} M_p(r, S_\varphi f)^p &= \frac{1}{2\pi}\int_0^{2\pi}\left|\varphi\big(f\big(re^{i\theta}\big)\big)\right|^p d\theta \\ &= \frac{1}{2\pi}p\int_0^\infty \lambda^{p-1}\left|\left\{\theta\in[0,2\pi]\colon \left|\varphi\big(f\big(re^{i\theta}\big)\big)\right| > \lambda\right\}\right| d\lambda. \end{aligned} \tag{13}$$

Now, if $|\varphi(f(re^{i\theta}))| > \lambda$, using (12) we see that

$$Be^{\varepsilon|f(re^{i\theta})|} > \lambda,$$

and consequently,

$$\left|f\big(re^{i\theta}\big)\right| > \frac{1}{\varepsilon}\log\frac{\lambda}{B}.$$

Using this in (13) we obtain

$$\begin{aligned} M_p(r, S_\varphi f)^p &\leqslant \frac{p}{2\pi}\int_0^\infty \lambda^{p-1}\left|\left\{\theta\in[0,2\pi]\colon \left|f\big(re^{i\theta}\big)\right| > \frac{1}{\varepsilon}\log\frac{\lambda}{B}\right\}\right| d\lambda \\ &= \frac{p}{2\pi}\int_0^B \lambda^{p-1}\left|\left\{\theta\in[0,2\pi]\colon \left|f\big(re^{i\theta}\big)\right| > \frac{1}{\varepsilon}\log\frac{\lambda}{B}\right\}\right| d\lambda \end{aligned} \tag{14}$$

$$+\frac{p}{2\pi}\int_B^\infty \lambda^{p-1}\left|\left\{\theta\in[0,2\pi]:\ \left|f\left(re^{i\theta}\right)\right|>\frac{1}{\varepsilon}\log\frac{\lambda}{B}\right\}\right|d\lambda. \tag{15}$$

Now observe that for $0<\lambda<B$, the number $\frac{1}{\varepsilon}\log\frac{\lambda}{B}$ is negative, so the set given in (14) is the entire interval $[0,2\pi]$, of measure 2π. On the other hand, $\frac{1}{\varepsilon}\log\frac{\lambda}{B}$ is positive if $\lambda>B$ and, hence, we can apply the John–Nirenberg theorem in (15) with the function f_r and the entire circle $\mathbb{T}$. Here, the mean of f_r over the interval $\mathbb{T}$ is

$$m_{\mathbb{T}}(f_r)=\frac{1}{2\pi}\int_0^{2\pi} f_r\left(e^{i\theta}\right)d\theta=\frac{1}{2\pi}\int_0^{2\pi} f\left(re^{i\theta}\right)d\theta=f(0)=0.$$

Using these facts and Lemma 3, we obtain

$$\begin{aligned}
M_p(r,S_\varphi f)^p &\leqslant p\int_0^B \lambda^{p-1}\,d\lambda+\frac{p}{2\pi}\int_B^\infty \lambda^{p-1}\left|\left\{\theta\in[0,2\pi]:\ \left|f_r\left(e^{i\theta}\right)-m_{\mathbb{T}}(f_r)\right|>\frac{1}{\varepsilon}\log\frac{\lambda}{B}\right\}\right|d\lambda\\
&\leqslant B^p+p\int_B^\infty \lambda^{p-1}K\exp\left(-\beta\frac{1}{\|f_r\|_*}\frac{1}{\varepsilon}\log\frac{\lambda}{B}\right)d\lambda\\
&\leqslant B^p+pK\int_B^\infty \lambda^{p-1}\exp\left(-\frac{\beta}{3\varepsilon\|f\|_*}\log\frac{\lambda}{B}\right)d\lambda.
\end{aligned}$$

Let us set $c=\frac{\beta}{3\varepsilon\|f\|_*}$, a positive constant which does not depend on r. Using (11) we see that $c>p$, and we can evaluate the last integral as follows

$$\int_B^\infty \lambda^{p-1}\exp\left(-c\log\frac{\lambda}{B}\right)d\lambda=\int_B^\infty \lambda^{p-1}\exp\left(\log\left(\frac{\lambda}{B}\right)^{-c}\right)d\lambda=\int_B^\infty \lambda^{p-1}\left(\frac{\lambda}{B}\right)^{-c}d\lambda=B^c\int_B^\infty \lambda^{p-1-c}\,d\lambda=\frac{B^p}{c-p}.$$

So we have proved that

$$M_p(r,S_\varphi f)^p\leqslant B^p+pK\frac{B^p}{c-p}=B^p\left(1+\frac{pK}{c-p}\right).$$

Here, the constant in the right-hand side does not depend on r. Then we conclude that $S_\varphi f\in H^p$. □

Proof of Theorem 1(b). Let $0<p<\infty$ and φ an entire function such that S_φ maps $\mathcal{B}$ into H^p. Take $f\in\mathcal{B}$, $f\not\equiv 0$, such that the sequence $\{a_n\}_{n=1}^\infty$ of its zeros, repeated according to multiplicity, does not satisfy the Blaschke condition, that is,

$$\sum_{n=1}^\infty\left(1-|a_n|\right)=\infty.$$

See, for example, Theorem 6 of [14] to check that such a function f exists.

Define $\varphi_1=\varphi-\varphi(0)$, and consider the function $g=\varphi_1\circ f$. Then g is analytic in the disc, and there are two possibilities:

- If $g\equiv 0$, then $\varphi_1(f(z))=0$, for every $z\in\mathbb{D}$. Now, $f(\mathbb{D})$ is an open set, since f is a non-constant analytic function in $\mathbb{D}$. Thus, φ_1 is an entire function that vanishes on an open set, which implies that $\varphi_1\equiv 0$ and φ is constant.
- If $g\not\equiv 0$, using that $S_\varphi f=\varphi\circ f\in H^p$, we see that g, given by

  $$g(z)=\varphi_1\left(f(z)\right)=\varphi\left(f(z)\right)-\varphi(0),\quad z\in\mathbb{D},$$

  also belongs to H^p. Then, the sequence $\{z_k\}$ of the zeros of g, repeated according to multiplicity, satisfy the Blaschke condition, $\sum(1-|z_k|)<\infty$. Now, for all n we have

  $$g(a_n)=\varphi\left(f(a_n)\right)-\varphi(0)=\varphi(0)-\varphi(0)=0,$$

  that is, every point of the sequence $\{a_n\}_{n=1}^\infty$ is a zero of g. Then

  $$\infty=\sum_{n=1}^\infty\left(1-|a_n|\right)\leqslant\sum_{k=1}^\infty\left(1-|z_k|\right)<\infty,$$

  which is a contradiction. Hence, this possibility cannot happen.

Consequently, φ must be a constant function. □

Proof of Theorem 1(c). Let $0 \leqslant s < \infty$ and φ an entire function such that $S_\varphi(Q_s) \subset H^\infty$. Suppose that φ is non-constant and we will get a contradiction.

If φ is non-constant then it is unbounded since it is entire. We can take a sequence $\{w_n\}_{n=1}^\infty$ such that

$$|w_1| > 1, \qquad |w_{n+1}| > 2|w_n|, \quad \text{for all } n \geqslant 1, \qquad \varphi(w_n) \to \infty, \quad \text{as } n \to \infty.$$

We may assume without loss of generality that all the points w_n of the sequence lie in the angular sector $A = \{w\colon |\arg w| < \frac{\pi}{8}\}$. Otherwise we can choose an angular sector of opening $\frac{\pi}{4}$ which contains infinitely many w_n and use the entire function $\varphi(e^{i\theta} z)$ instead of $\varphi(z)$, where $z \mapsto e^{i\theta} z$ maps A onto that sector.

For each $n = 1, 2, \dots,$ take a number d_n such that $\frac{1}{|w_{n+2}|^2} < d_n < \frac{1}{|w_{n+1}|^2}$ and consider the disc

$$D_n = \{z \in \mathbb{C}\colon |z - w_n| < d_n\}.$$

Observe that $|w_1| < |w_2| < \cdots$ and $|w_n| > 2^{n-1}$, for $n = 1, 2, \dots,$ thus $\{d_n\}$ decrease to zero as $n \to \infty$. Now, for each $n = 1, 2, \dots,$ let G_n be the convex hull of $D_n \cup D_{n+1}$. Denoting $|G_n|$ the area of G_n, we see that

$$\begin{aligned}
|G_n| &\leqslant 2d_n\bigl(d_n + |w_n - w_{n+1}| + d_{n+1}\bigr) \\
&\leqslant 2d_n\bigl(2d_n + \bigl(|w_n| + |w_{n+1}|\bigr)\bigr) \\
&\leqslant \frac{2}{|w_{n+1}|^2}\left(\frac{2}{|w_{n+1}|^2} + 2|w_{n+1}|\right) \\
&\leqslant \frac{2}{|w_{n+1}|^2}\bigl(2|w_{n+1}| + 2|w_{n+1}|\bigr) \\
&\leqslant \frac{8}{|w_{n+1}|} < \frac{8}{2^n}.
\end{aligned}$$

Define

$$G = \bigcup_{n=1}^\infty G_n.$$

It is easy to check that G is a simply connected domain that contains all the points w_n. We have that

$$|G| \leqslant \sum_{n=1}^\infty |G_n| \leqslant \sum_{n=1}^\infty \frac{8}{2^n} < \infty.$$

Let f be a Riemann map of $\mathbb{D}$ onto G. Since G has finite area, $f \in \mathcal{D}$ and, hence, $f \in Q_s$. Then $S_\varphi f \in H^\infty$. For every $n = 1, 2 \dots,$ let z_n be the point in $\mathbb{D}$ for which $f(z_n) = w_n$. Then

$$(S_\varphi f)(z_n) = \varphi\bigl(f(z_n)\bigr) = \varphi(w_n) \to \infty, \quad \text{as } n \to \infty,$$

which leads us to contradiction and finishes the proof. □

Remark 1. In the proof of part (b) of Theorem 1, we used that the sequence of zeros of a function in H^p satisfies the Blaschke condition, property that is also true for the Nevanlinna class $\mathcal{N}$. Thus, we see that the Hardy space H^p may be replaced by the wider class $\mathcal{N}$. That is, we have:

Let φ be an entire function. Then $S_\varphi(\mathcal{B}) \subset \mathcal{N}$ if and only if φ is constant.

4. Proof of Theorem 2

Proof of Theorem 2(a). Bearing in mind that if $0 < p < \infty$ and $0 < \alpha < \frac{1}{p}$ then the function

$$f(z) = \frac{1}{(1-z)^\alpha}, \quad z \in \mathbb{D},$$

belongs to H^p, and arguing as in the proof of Proposition 1 of [1], we see that, if $0 < p < \infty$ then $S_\varphi(H^p) \subset \mathcal{B}$ if and only if φ is constant. Since $Q_s \subset \mathcal{B}$ for all $s \in [0, \infty)$, this implies that, if $0 < p < \infty$ and $0 \leqslant s < \infty$ then $S_\varphi(H^p) \subset Q_s$ if and only if φ is constant. □

Proof of Theorem 2(b). Let $0 \leqslant s < 1$ and suppose that φ is an entire function satisfying $S_\varphi(H^\infty) \subset Q_s$. Let us assume that φ is not constant to get a contradiction. If φ is not constant then $\varphi' \not\equiv 0$ and there exist a complex number z_0 and positive constants R, A such that

$$\left|\varphi'(z)\right| \geqslant A, \quad \text{if } |z-z_0| \leqslant R. \tag{16}$$

Now we use the fact that $H^\infty \setminus Q_s \neq \emptyset$ to take a function $f \in H^\infty$ such that $f \notin Q_s$. For example, using the characterization of power series with Hadamard gaps in Q_s proved by Aulaskari, Xiao and Zhao in [4], we easily deduce that the function f defined by

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{k^2} z^{2^k}, \quad z \in \mathbb{D},$$

does not belong to Q_s (see also [12, pp. 192–193]). On the other hand, since $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, we have trivially that $f \in H^\infty$.

Now, since $f \in H^\infty$ we can take $M > 0$ such that $|f(z)| \leqslant M$ for any $z \in \mathbb{D}$. Consider the function h defined by

$$h(z) = z_0 + \frac{R}{M} f(z), \quad z \in \mathbb{D}.$$

Then $h \in H^\infty \setminus Q_s$, and $|h(z) - z_0| \leqslant R$, for any $z \in \mathbb{D}$, so applying (16) we have that

$$\left|\varphi'\big(h(z)\big)\right| \geqslant A, \quad \text{for all } z \in \mathbb{D}. \tag{17}$$

Using the assumption that $S_\varphi(H^\infty) \subset Q_s$, we deduce that $S_\varphi(h) = \varphi \circ h$ belongs to Q_s, that is,

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left|(\varphi \circ h)'(z)\right|^2 g(z,a)^s \, dx\, dy < \infty. \tag{18}$$

Now, for any $a \in \mathbb{D}$, using (17) we see that

$$\begin{aligned}\int_{\mathbb{D}} \left|(\varphi \circ h)'(z)\right|^2 g(z,a)^s \, dx\, dy &= \int_{\mathbb{D}} \left|h'(z)\right|^2 \left|\varphi'\big(h(z)\big)\right|^2 g(z,a)^s \, dx\, dy \\ &\geqslant A^2 \int_{\mathbb{D}} \left|h'(z)\right|^2 g(z,a)^s \, dx\, dy. \end{aligned} \tag{19}$$

From (18) and (19) it follows that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left|h'(z)\right|^2 g(z,a)^s \, dx\, dy < \infty,$$

which means that $h \in Q_s$ and leads us to contradiction. □

Proof of Theorem 2(c). It is clear that $S_\varphi(H^\infty) \subset H^\infty$ for any entire function φ. Since $H^\infty \subset BMOA = Q_1$ and $H^\infty \subset \mathcal{B} = Q_s$ for all $s > 1$, we deduce that $S_\varphi(H^\infty) \subset Q_s$ for all s, $1 \leqslant s < \infty$, and for any entire function φ. □

We remark that the analogue question of characterizing the superposition operators between Q_s spaces and Bergman spaces is essentially known. Recall that, if $0 < p < \infty$, the Bergman space A^p is the set of all analytic functions f in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} \left|f(z)\right|^p \, dx\, dy < \infty.$$

Theorem C. *Let φ be an entire function. Then*

(a) *For $0 < p < \infty$, $S_\varphi(\mathcal{D}) \subset A^p$ if and only if φ is of order less than 2, or of order 2 and finite type.*
(b) *For $0 < s < \infty$ and $0 < p < \infty$, $S_\varphi(Q_s) \subset A^p$ if and only if φ is of order less than one, or of order one and type zero.*
(c) *For $0 < p < \infty$ and $0 \leqslant s < \infty$, $S_\varphi(A^p) \subset Q_s$ if and only if φ is constant.*

Indeed, part (a) was proved by Buckley and Vukotić in [8] (see Theorem 1 for $p = 2$). Part (b) was proved in [1] for the Bloch space ($s > 1$). Now, looking at the proof of (a) ⇒ (c) in Theorem 3, we observe that the Bloch function used there is univalent, then it also belongs to the space Q_s, $0 < s \leqslant 1$, as we mentioned before. Then the same proof works to check that if $S_\varphi(Q_s) \subset A^p$ ($0 < s < \infty$, $0 < p < \infty$) then φ is of order less than one, or of order one and type zero. The other implication follows also from Theorem 3 of [1], bearing in mind that $Q_s \subset \mathcal{B}$ for all s.

Finally, we easily deduce part (c) from Proposition 1 of [1], using again the inclusions $Q_s \subset \mathcal{B}$ for all s.

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